Toric geometry, exercise sheet 1

Exercise 1. For each of the following affine toric varieties from Example 2.9, find a suitable lattice M and finite subset $\mathcal{A} \subseteq M$ such that $Y_{\mathcal{A}}$ is the given variety. Write down $\Phi_{\mathcal{A}}$ and justify your answer.

(a) The torus $T = (\mathbb{C}^*)^n$.
(b) Affine space $\mathbb{C}^n$.
(c) The cuspidal cubic $V(y^2 - x^3) \subseteq \mathbb{C}^2$.
(d) The threefold node $V(xy - zw) \subseteq \mathbb{C}^4$.

Exercise 2. Let $d \geq 2$ and choose $\mathcal{A} = \{(0,1), (1,1), \ldots, (d,1)\} \subseteq \mathbb{Z}^2$.

(a) Show that $I(Y_{\mathcal{A}})$ is generated by the 2×2 minors of

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{d-1} \\ x_1 & x_2 & \cdots & x_d \end{pmatrix}.$$

In other words, prove

$$I(Y_{\mathcal{A}}) = (x_i x_{j+1} - x_{i+1} x_j \mid 0 \leq i < j \leq d-1).$$

(b) Consider the alternative set

$$\mathcal{B} = \{(d,0), (d-1,1), \ldots, (0,d)\} \subseteq \mathbb{Z}^2.$$

Determine $\mathbb{Z}\mathcal{A}$ and $\mathbb{Z}\mathcal{B}$. Find an injective lattice homomorphism $L : \mathbb{Z}^2 \to \mathbb{Z}^2$ with $L(\mathcal{A}) = \mathcal{B}$, and calculate the index of its image.

(c) Show that $Y_{\mathcal{A}} = Y_{\mathcal{B}}$ by comparing the monomial maps.

(d) Prove that $Y_{\mathcal{A}}$ is also the image of

$$\psi : \mathbb{C}^2 \longrightarrow \mathbb{C}^{d+1}, \qquad (u,v) \longmapsto (u^d, u^{d-1}v, \ldots, uv^{d-1}, v^d).$$

Conclude that $Y_{\mathcal{A}}$ is the affine cone over the degree-d rational normal curve

$$C_d = \{[u^d : u^{d-1}v : \cdots : v^d] \mid [u:v] \in \mathbb{P}^1\} \subseteq \mathbb{P}^d.$$

Exercise 3. Prove the following lemmas.

Lemma 2.13. Let $\varphi : T_1 \to T_2$ be a morphism of tori. The image $\varphi(T_1)$ is a closed subtorus of T_2. *Hint: Use the Smith normal form for matrices with entries in a principal ideal domain (like $\mathbb{Z}$).*

Lemma 2.11. Let $\varphi : T_1 \to T_2$ be a morphism of tori. Then there is an induced group homomorphism of character lattices

$$\widehat{\varphi} : M(T_2) \longrightarrow M(T_1), \qquad \chi^{\widehat{\varphi}(m)} = \chi^m \circ \varphi.$$

Moreover,

$$\varphi \text{ injective} \iff \widehat{\varphi} \text{ surjective},$$
$$\varphi \text{ surjective} \iff \widehat{\varphi} \text{ injective}.$$

Lemma 2.12. Let T be a torus with character lattice M. Let $\mathcal{A} = \{m_1, \ldots, m_s\} \subseteq M$. Lemma 2.11 applied to the morphism of tori $\Phi_{\mathcal{A}} : T \to (\mathbb{C}^*)^s$ induces the group homomorphism of character lattices given by

$$\widehat{\Phi}_{\mathcal{A}} : \mathbb{Z}^s \longrightarrow M, \qquad (a_1, \ldots, a_s) \longmapsto \sum_{i=1}^{s} a_i m_i.$$