Toric geometry, exercise sheet 2

By *cone* we always mean a cone.

Exercise 1. For the affine toric varieties in Example 2.9, find a lattice N and a cone $\sigma \subseteq N_{\mathbb{R}}$ with U_σ isomorphic to the given variety, or explain why no such cone exists. *Hint: We will see that U_σ is always a normal variety.*

(a) The torus $(\mathbb{C}^*)^n$.
(b) Affine space $\mathbb{C}^n$.
(c) The cuspidal cubic $V(y^2 - x^3) \subseteq \mathbb{C}^2$.
(d) The threefold node $V(xy - zw) \subseteq \mathbb{C}^4$.

Exercise 2. Let $\sigma_i \subseteq (N_i)_{\mathbb{R}}$ be cones for $i = 1, 2$. Their product is the cone

$$\sigma_1 \times \sigma_2 = \{(v_1, v_2) \in (N_1 \oplus N_2)_{\mathbb{R}} : v_i \in \sigma_i\}.$$

Prove that

$$U_{\sigma_1 \times \sigma_2} \cong U_{\sigma_1} \times U_{\sigma_2}.$$

Exercise 3. For $n = 1, 2$, classify all cones σ in $\mathbb{R}^n$ up to $\mathrm{GL}_n(\mathbb{Z})$-equivalence. Compute their dual cones $\sigma^\vee$ and say as much as you can about the affine toric variety U_σ.

Exercise 4. Prove part (3) of the following proposition, using parts (1) and (2).

Proposition 2.33. Let $\sigma \subseteq N_{\mathbb{R}}$ be a cone. Then

(1) $\sigma^\vee \subseteq M_{\mathbb{R}}$ is a cone.
(2) $(\sigma^\vee)^\vee = \sigma$ (identifying $N_{\mathbb{R}}$ with its double dual).
(3) For $m_1, \ldots, m_s \in M$,

$$\sigma^\vee = \mathrm{Cone}(m_1, \ldots, m_s) \quad \Longleftrightarrow \quad \sigma = H^+_{m_1} \cap \cdots \cap H^+_{m_s}.$$

Remark: In part (1), finite generation follows from the rational Minkowski-Weyl theorem. In part (2), the inclusion $(\sigma^\vee)^\vee \subseteq \sigma$ follows from the separating hyperplane theorem.

Exercise 5. Prove

Proposition 2.34. Let $\sigma \subseteq N_{\mathbb{R}}$ be a cone. Then

(0) If $\tau \preceq \sigma$, then τ is a cone.
(1) If $\tau, \tau' \preceq \sigma$, then $\tau \cap \tau' \preceq \sigma$.
(2) If $\tau \preceq \sigma$ and $\tau' \preceq \tau$, then $\tau' \preceq \sigma$.
(3) If $\tau \preceq \sigma$ and $v, w \in \sigma$, then $v + w \in \tau$ implies $v, w \in \tau$.
(4) There is an inclusion-reversing bijection

$$\{\text{faces of } \sigma\} \longleftrightarrow \{\text{faces of } \sigma^\vee\}.$$

Describe the correspondence explicitly.