Toric geometry, exercise sheet 4

Exercise 1. For each of the following finite sets $A \subset \mathbb{Z}^n$, determine the ideal and the singular locus of the affine toric variety $Y_A \subset \mathbb{A}^{|A|}$ and of the projective toric variety $X_A \subset \mathbb{P}^{|A|-1}$.

(a) $A = \{0, 1\} \subset \mathbb{Z}$.
(b) $A = \{2, 3\} \subset \mathbb{Z}$.
(c) $A = \{1, 2, 3\} \subset \mathbb{Z}$.
(d) $A = \{0, 1, 3\} \subset \mathbb{Z}$.
(e) $A = \{(1, 0), (0, 1), (1, 1), (2, 2)\} \subset \mathbb{Z}^2$.
(f) $A = \{0, 1, 2, 3\} \subset \mathbb{Z}$.
(g) $A = \{(0, 0), (2, 0), (0, 2), (1, 1)\} \subset \mathbb{Z}^2$.

Exercise 2. Take $M = \mathbb{Z}^2$ and $A = \{(1, 0), (0, 1), (1, 2), (2, 1)\}$. Prove that $\mathbb{Z}'A$ and $\mathbb{Z}A$ have equal rank, whereas $[\mathbb{Z}A : \mathbb{Z}'A] = 2$. Explain why the projective monomial map $\pi \circ \Phi_A$ is two-to-one.

Exercise 3. Let $M = \mathbb{Z}^2$ and $A = \{(0, 0), (1, 0), (0, 1), (1, 1)\} \subset M$. Consider the projective toric variety $X_A \subset \mathbb{P}^3$.

(a) Write down the projective monomial map defining X_A and determine its homogeneous ideal.
(b) Show that X_A is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ by giving an explicit map.
(c) Describe the action of $(\mathbb{C}^*)^2$ on X_A and find its fixed points.

Exercise 4. Prove the following proposition. Let $A = \{m_1, \ldots, m_s\} \subset \mathbb{Z}^n$ and $\widehat{A} = \{(m_1, 1), \ldots, (m_s, 1)\} \subset \mathbb{Z}^{n+1}$.

Proposition 3.7. $X_A = X_{\widehat{A}}$ and $Y_{\widehat{A}}$ is the affine cone over $X_{\widehat{A}}$.