Toric geometry, exercise sheet 7

Exercise 1. Let Σ be a fan in $N_{\mathbb{R}}$ and write $V(\sigma) = \overline{O(\sigma)}$. Prove that

$$\operatorname{Sing}(X_\Sigma) = \bigcup_{\substack{\sigma \in \Sigma \\ \sigma \text{ not smooth}}} V(\sigma).$$

Exercise 2. Let Σ be the complete fan in $N_{\mathbb{R}} = \mathbb{R}^3$, where $N = \mathbb{Z}^3$, with rays generated by

$$v_1 = (1,0,0), \quad v_2 = (0,1,0), \quad v_3 = (0,0,1), \quad v_4 = (-1,-2,-2),$$

and maximal cones generated by any three of these vectors.

(a) Show that $X_\Sigma \cong \mathbb{P}(1,1,2,2)$.

(b) Determine all torus orbits and the isomorphism types of their closures. In particular, show that precisely two of the two-dimensional orbit closures are isomorphic to $\mathbb{P}^2$.

(c) Determine all non-smooth cones of Σ and compute $\operatorname{Sing}(X_\Sigma)$.

Exercise 3. Let $N = \mathbb{Z}^3$ and consider the cone

$$\sigma = \operatorname{Cone}\Big((1,0,0),(1,2,0),(1,0,2)\Big) \subset N_{\mathbb{R}}.$$

(a) Compute $\sigma^\vee$ and the Hilbert basis of $S_\sigma = \sigma^\vee \cap M$. Show $U_\sigma \cong Z(t^2 - xyz) \subset \mathbb{C}^4$.

(b) Determine all torus orbits and their closures in the coordinates x, y, z, t. Show that every proper orbit closure is smooth.

(c) Compute the singular locus of U_σ directly from its equation and compare your result with Exercise 1.

Exercise 4. Show that $\operatorname{Star}(\sigma)$ is a fan in $N(\sigma)_{\mathbb{R}}$ and $V(\sigma) \cong X_{\operatorname{Star}(\sigma)}$ as toric varieties.