THETA FUNCTIONS, BROKEN LINES AND 2-MARKED LOG GROMOV-WITTEN INVARIANTS

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Abstract. Theta functions were defined for varieties with effective anticanonical divisor [GHS] and are related to certain punctured Gromov-Witten invariants [ACGS2]. In this paper we show that in the case of a log Calabi-Yau surface (X, D) with smooth very ample anticanonical divisor we can circumvent the notion of punctured Gromov-Witten invariants and relate theta functions and their multiplicative structure to certain 2-marked log Gromov-Witten invariants. This is a natural extension of the correspondence between wall functions and 1-marked log Gromov-Witten invariants [Grä].

Contents

Introduction

Classically theta functions were defined in the theory of elliptic curves. Mumford generalized their definition to Abelian varieties and studied degenerations of such varieties. In [GHK] theta functions were used to construct the mirror to certain maximally degenerated log Calabi-Yau pairs. They natrually fit within the Gross-Siebert program of mirror symmetry [GS1][GS2][GS3], where a generalization of Mumford's degeneration is considered [GHS]. It was shown in [GS7], Theorem 4.5, that these theta functions are linked to certain *punctured Gromov-Witten invariats* [ACGS2]. This leads to the *intrinsic mirror symmetry* construction [GS6], where the mirror to a Calabi-Yau variety or log Calabi-Yau pair is constructed from its theta functions or, equivalently, its punctured invariants.

In this paper we consider the case complementary to [GHK] of a log Calabi-Yau pair (X, D) with D a *smooth* divisor. We restrict to the case of a variety X with very ample anticanonical bundle to obtain a more combinatorial description.

Moreover, we restrict to the 2-dimensional case. We show that in this case we can circumvent the notion of punctured invariants and relate theta functions to certain 2-marked *log Gromov-Witten invariants*.

Definition. For an effective curve class β on X and $p,q\in\mathbb{Z}_{>0}$ with $p+q=D\cdot\beta$ let $\beta_{p,q}$ be the class of stable log maps of class β with two marked points of contact orders p and q with D. Let $\mathscr{M}(X,\beta_{p,q})$ be the moduli space of basic stable log maps of class $\beta_{p,q}$. By [GS5] this a proper Deligne-Mumford stack of virtual dimension 1 and admits a virtual fundamental class $[\![\mathscr{M}(X,\beta_{p,q})]\!]$. Let $\text{ev}:\mathscr{M}(X,\beta_{p,q})\rightarrow D$ be the evaluation map at the marked point of order p. Define the 2-marked log Gromov-Witten invariant

$$N_{p,q}(X,\beta):=\int_{[\![\mathscr{M}(X,\beta_{p,q})]\!]}\text{ev}^\star[\text{pt}]\in\mathbb{Q}$$

and write

$$N_{p,q}(X):=\sum_\beta N_{p,q}(X,\beta),$$

where the sum is over all effective curve classes of X. We will omit X in the notation whenever it is clear from the context.

For an asymptotic exponent m and a point P on the dual intersection complex $(B,\mathscr{P},\varphi)$ of X, there is a theta function defined by

$$\vartheta_m(P)=\sum_{\mathfrak{b}\in\mathfrak{B}_m(P)}a_{\mathfrak{b}}z^{m_{\mathfrak{b}}}$$

The sum is over *broken lines* – piecewise linear maps $\mathfrak{b}:[0,\infty)\rightarrow B$ with a monomial $a_iz^{m_i}$ attached to every linear segment such that $m_i=(\bar{m}_i,h_i)$ with $\bar{m}_i$ parallel to the segment and $a_iz^{m_i}$, $a_{i+1}z^{m_{i+1}}$ are related by wall crossing in $\mathscr{S}$.

By [GHS], Theorem 3.24, theta functions generate a ring with multiplication rule

$$\vartheta_p(P)\cdot\vartheta_q(P)=\sum_{r=0}^\infty\alpha_{p,q}^r(P)\vartheta_r(P)$$

with structure constants

$$\alpha_{p,q}^r(P)=\sum_{\substack{(\mathfrak{b}_1,\mathfrak{b}_2)\in\mathfrak{B}_p(P)\times\mathfrak{B}_q(P)\\ m_{\mathfrak{b}_1}+m_{\mathfrak{b}_2}=\ m}}a_{\mathfrak{b}_1}a_{\mathfrak{b}_2}.$$

Let P be a point infinitely far away from the bounded maximal cell of B. Let m_{out} be the unique unbounded direction of B and write $x:=z^{(-m_{\text{out}},-1)}$ and $t:=z^{(0,1)}$. Note that at P we have $\varphi(-m_{\text{out}})=-1$, so x has t-order zero.

Theorem 1. *We have*

$$\vartheta_q(P)=x^{-q}+\sum_{p=1}^\infty pN_{p,q}x^pt^{p+q}.$$

The reason for this equation is as follows. As we move P away from the bounded cell the slope of the walls (meaning their scalar product with the unbounded direction) increases. If we move sufficiently far away all broken lines ending in P have to be parallel to the unbounded direction (Proposition 2.4). We can complete such a broken line to a tropical curve with two unbounded legs. One of these legs contains P, corresponding to a fixed point. Then the tropical correspondence theorem for log Calabi-Yau pairs with smooth divisor [Grä] (more precisely, an extension of it incluing point conditions) gives a relation between broken lines and 2-marked log Gromov-Witten invariants, leading to the above correspondence for theta functions. By a similar reason we have the following.

Theorem 2. *The multiplication rule for theta functions is given by*

$$\vartheta_p \cdot \vartheta_q = \vartheta_{p+q} + \sum_{r=0}^{max\{p,q\}-1} \alpha_{p,q}^r \vartheta_r$$

with structure constants

$$\alpha_{p,q}^r = (p-r)N_{p-r,q} + (q-r)N_{q-r,p}.$$

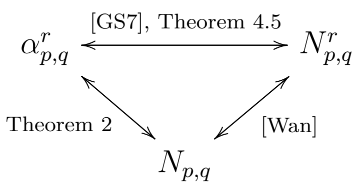
Here we define $N_{p,q}(X) = 0$ when $p \leq 0$. Comparing this with Theorem 1 we obtain relations between the numbers $N_{p,q}$. These determine all $N_{p,q}$ from the invariants $N_{1,q}$ with fixed point multiplicity 1.

Remark. The invariants $N_{1,q}(X,\beta)$ agree with 1-marked invariants $N_q(\hat{X}, \pi^\star\beta - C)$ of the blow up at a point $\pi : \hat{X} \to X$, where C is the exceptional divisor. This can be shown via degeneration to the normal cone ([GRZ], Proposition 5.2). Using this relation, Theorem 2 above, the log-local correspondence of [GGR] and results of [CLL] we show in [GRZ] that the proper Landau-Ginzburg model ϑ_1 agrees with the open mirror map.

Remark. In [GS6] the structure constants were defined as

$$\alpha_{p,q}^r = \sum_\beta N_{pq}^r(\beta) t^\beta$$

where the sum is over all effective curve classes β of X and $N_{pq}^r(\beta)$ are certain punctured Gromov-Witten invariants of class β. In particular α_{m_1,m_2}^m does not depend on P. It was shown in [GS7], Theorem 4.5, that this agrees with the above definition of $\alpha_{p,q}^r(P)$. Yu Wang [Wan] is proving a relation between punctured and 2-marked invariants, leading to a triangle of relations



1. Toric degenerations and wall structures

Tropical geometry is a piecewise linear version of algebraic geometry, hence naturally linked to combinatorics. The most natural class of varieties on which to consider tropical geometry are toric varieties. They admit a combinatorial description in terms of the orbits of their torus action. This can be made explicit in terms of a fan or, dually and given a polarization, a "momentum" polytope. A similar combinatorial description for a non-toric variety X is obtained via a toric degeneration $\mathfrak{X}$ – a degeneration whose central fiber X_0 is a union of toric varieties glued along toric divisors and such that the family is strictly semistable away from a codimension 2 subset.

The dual intersection complex $(B,\mathscr{P})$ of $\mathfrak{X}$ is obtained by gluing together the fans of the irreducible components of X_0 according to the combinatorics of their intersection. Locally at a vertex such a fan induces an affine structure on B. An affine structure on maximal cells is given by the local structure of the family $\mathfrak{X}$ at the corresponding point. This is an affine toric variety defined by a cone over a polytope, and this polytope gives the affine structure on the maximal cell. These affine structures need not fit together, so B is an affine manifolds with singularities. It is glued from (possibly unbounded) polytopes and $\mathscr{P}$ is a collection of these. A polarization of X corresponds to a strictly convex piecewise affine function φ on $(B,\mathscr{P})$.

Similarly, the intersection complex $(\check{B},\check{\mathscr{P}},\check{\varphi})$ is obtained by gluing together momentum polytopes of the irreducible components of X_0. To do so, we need a polarization on X. In this case the piecewise affine function $\check{\varphi}$ describes the family $\mathfrak{X}$ locally: it is the upper convex hull of $\check{\varphi}$.

Example 1.1. A toric degeneration of $(\mathbb{P}^2,E)$, for E an elliptic curve, is given by

$$\begin{aligned} \mathfrak{X} &= \{XYZ = t^3(W+f_3)\} \subset \mathbb{P}(1,1,1,3)\times\mathbb{A}^1 \rightarrow \mathbb{A}^1, \\ \mathfrak{D} &= \{W=0\} \subset \mathfrak{X}. \end{aligned}$$

Here W is the variable in $\mathbb{P}(1,1,1,3)$ of weight 3, t is the variable of $\mathbb{A}^1$ and the map is given by projection to t. Moreover, f_3 is a polynomial in X,Y,Z, general in the sense that $XYZ/t^3 - f_3$ is nonsingular for general t.

Indeed, for $t\neq 0$ we can resolve for $W = XYZ/t^3 - f_3$, so the general fiber of $\mathfrak{X}$ is $X=\mathbb{P}^2$ and the general fiber of $\mathfrak{D}$ is an elliptic curve E by the generality condition on f_3. For $t=0$ we have $XYZ=0$, so X_0 is a union of three $\mathbb{P}(1,1,3)$ glued along toric divisors. Figure 1.1 shows the intersection complex and its dual for this toric degeneration.

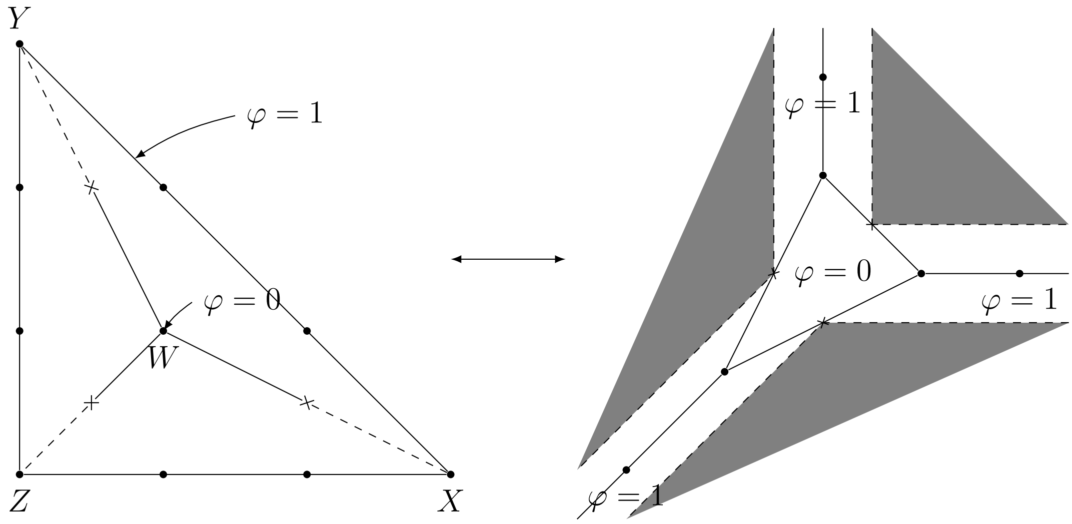


FIGURE 1.1. The intersection complex (left) of $(\mathbb{P}^2, E)$ and its dual (right).

It is natural to ask whether one can go back and construct a toric degeneration $\check{\mathfrak{X}}$ from a triple $(B, \mathscr{P}, \varphi)$ as above such that $(B, \mathscr{P}, \varphi)$ is the intersection complex of $\check{\mathfrak{X}}$. This was solved under some assumptions in [GS3] and involves the iterative construction of wall structure via scattering calculations. A wall is a polyhedral subset $\mathfrak{p}$ of B with some attached function $f_{\mathfrak{p}}$ living in the ring N_φ defined by

$$\begin{aligned} P_\varphi &= \{m = (\bar{m}, h) \in M \oplus \mathbb{Z} \mid h \geq \varphi(\bar{m})\}, \\ N_\varphi &= \varprojlim \mathbb{C}[P_\varphi]/(t^k), \quad t = z^{(0,1)}. \end{aligned}$$

The initial wall structure $\mathscr{S}_0$ consists of walls coming out of the affine singularities of B with attached functions $1 + z^{(\bar{m},0)}$, where $\bar{m}$ is the direction of $\mathfrak{p}$. A wall defines an automorphism of some localization of N_φ by

$$\theta_{\mathfrak{p}}(z^{(\bar{m},h)}) = f_{\mathfrak{p}}^{-\langle n,\bar{m}\rangle} z^{(\bar{m},h)}$$

Scattering means whenever two or more walls intersect, we introduce more rays with base at the intersection point, such that the clockwise composition of the automorphisms $\theta_{\mathfrak{p}}$ is the identity. For any finite t-order k this produces finitely many new rays, leading to a wall structure $\mathscr{S}_k$ that is "consistent to order k". The formal limit for $k \to \infty$ is denoted by $\mathscr{S}_\infty$. Note that the t-order of $z^{(\bar{m},h)}$ at x is $\varphi_x(-\bar{m}) + h \geq 0$, where φ_x is a local representative of φ at x, since

$$z^{(\bar{m},h)} = \left(z^{(-\bar{m},\varphi_x(-\bar{m}))}\right)^{-1} t^{\varphi_x(-\bar{m})+h}.$$

Example 1.2. Figure 1.2 shows the wall structure consistent to t-order $2 \cdot 3 = 6$ for $(\mathbb{P}^2, E)$ in two different charts. There are functions attached to each of these walls, but we don't show them for readability.

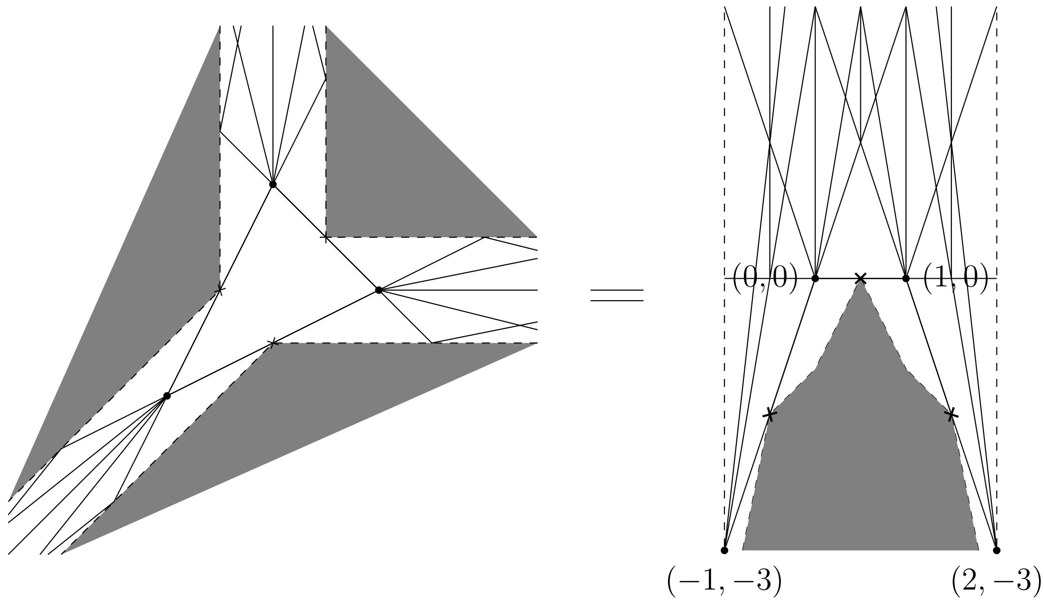


FIGURE 1.2. The wall structure for $(\mathbb{P}^2, E)$ of order $2 \cdot 3 = 6$.

The idea of the Gross-Siebert program is that the family $\check{\mathfrak{X}}$ constructed this way is a degeneration of the mirror to (X, D). As is well known, the mirror to a log Calabi-Yau pair (X, D) should be a Landau-Ginzburg model, that is, a toric variety Y together with a "superpotential" $W : Y \to \mathbb{C}$ whose critical locus is compact. It is shown in [CPS] that such a superpotential can be constructed from broken lines, see §2. For more details on these constructions see the original papers [GS1][GS2][GS3] and the introductory paper [GS4].

2. Broken lines

Definition 2.1. A *broken line* for a wall structure $\mathscr{S}$ on $(B, \mathscr{P})$ is a proper continuous map

$$\mathfrak{b} : (-\infty, 0] \to B_0$$

with image disjoint from any joints of $\mathscr{S}$, along with a sequence $-\infty = t_0 < t_1 < \cdots < t_r = 0$ for some $r \geq 1$ with $\mathfrak{b}(t_i) \in |\mathscr{S}|$ for $i \leq r - 1$, and for each $i = 1, \ldots, r$ an expression $a_i z^{m_i}$ with $a_i \in \mathbb{C} \setminus \{0\}$, $\bar{m}_i \in \Lambda_{\mathfrak{b}(t)}$ for any $t \in (t_{i-1}, t_i)$, defined at all points of $\mathfrak{b}([t_{i-1}, t_i])$, and subject to the following conditions:

(1) $\mathfrak{b}|_{(t_{i-1}, t_i)}$ is a non-constant affine map with image disjoint from $|\mathscr{S}|$, hence contained in the interior of a unique chamber $\mathfrak{u}_i$ of $\mathscr{S}$, and $\mathfrak{b}'(t) = -m_i$ for all $t \in (t_{i-1}, t_i)$.

(2) For each $i = 1, \ldots, r - 1$ the expression $a_{i+1} z^{m_{i+1}}$ is a result of transport of $a_i z^{m_i}$ from $\mathfrak{u}_i$ to $\mathfrak{u}_{i+1}$, i.e., is a term in the expansion of $\theta_{\mathfrak{u}_i \mathfrak{u}_{i+1}}(a_i z^{m_i})$.

(3) $a_1 = 1$ and $m_1 = (\bar{m}_1, h)$ has t-order zero at $\mathfrak{b}(t_1)$, i.e., $h = \varphi(\bar{m}_1)$.

Write $a_{\mathfrak{b}} z^{m_{\mathfrak{b}}}$ for the ending monomial $a_r z^{m_r}$.

The next proposition shows that the breaking of a broken line at a wall is similar to the scattering of two walls.

Proposition 2.2 (Scattering of monomials)**.** *Let $\mathfrak{p}$ be a wall and let $\mathfrak{b}$ be a broken line intersecting $\mathfrak{p}$ with local monomial az^m. Consider the scattering diagram $\mathfrak{D}$ consisting of two lines $\mathfrak{d}_\mathfrak{p} = (\mathbb{R}\bar{m}_\mathfrak{p}, f_\mathfrak{p})$ and $\mathfrak{d}_\mathfrak{b} = (\mathbb{R}\bar{m}, 1 + az^m)$. Then breakings of $\mathfrak{b}$, i.e., terms in $\theta_\mathfrak{p}(az^m) = \sum_i a_i z^{m_i}$ correspond to rays in $\mathfrak{D}_\infty$ with direction $\bar{m}_i = k\bar{m}_\mathfrak{p} + \bar{m}$ for some $k \geq 0$. The corresponding ray function is $1 + a_i z^{m_i}$.*

Proof. Let γ be a simple counterclockwise loop around the origin starting in the cone $\mathbb{R}_{>0}m_\mathfrak{p} + \mathbb{R}_{>0}m$. We show that $\theta_{\gamma,\mathfrak{D}}\theta_{\mathfrak{d}_\mathfrak{b}}$ equals $1 + \theta_\mathfrak{p}(az^m)\partial_n$ modulo $(z^m)^2$. Working modulo $(z^m)^2$ it is enough to consider $\theta_{\mathfrak{d}_\mathfrak{b}}$ to first order:

$$\theta_{\mathfrak{d}_\mathfrak{b}} = 1 + az^m\partial_n \bmod (z^m)^2$$

where n is the primitive normal vector to m pointing in the direction of γ and ∂_n is the automorphism of N_φ defined by $\partial_n : z^{m'} \mapsto \partial_n z^{m'} = \langle n, m' \rangle z^m$. Then

$$\begin{aligned} \theta_{\gamma,\mathfrak{D}}\theta_{\mathfrak{d}_\mathfrak{b}} = \theta_{\mathfrak{d}_\mathfrak{p}}\theta_{\mathfrak{d}_\mathfrak{b}}\theta_{\mathfrak{d}_\mathfrak{p}}^{-1} &\equiv \theta_{\mathfrak{d}_\mathfrak{p}}(1 + az^m\partial_n)\theta_{\mathfrak{d}_\mathfrak{p}}^{-1} \bmod (z^m)^2 \\ &\equiv \theta_{\mathfrak{d}_\mathfrak{p}}\theta_{\mathfrak{d}_\mathfrak{p}}^{-1} + \theta_{\mathfrak{d}_\mathfrak{p}}(az^m)\partial_n\theta_{\mathfrak{d}_\mathfrak{p}}\theta_{\mathfrak{d}_\mathfrak{p}}^{-1} \bmod (z^m)^2 \\ &\equiv 1 + \theta_{\mathfrak{d}_\mathfrak{p}}(az^m)\partial_n \bmod (z^m)^2 \end{aligned}$$

By the definition of scattering each term of $\theta_{\mathfrak{d}_\mathfrak{p}}(az^m)$ gives a wall in $(\mathfrak{D}_\infty \setminus \mathfrak{D}) \cup \{\mathfrak{d}_\mathfrak{b}^+\}$ where $\mathfrak{d}_\mathfrak{b}^+$ is the outgoing wall of the line $\mathfrak{d}_\mathfrak{b}$. □

Let $(B, \mathscr{P}, \varphi)$ be the dual intersection complex of a pair (X, D) consisting of a surface X and a smooth very ample anticanonical divisor D. Then B has one unbounded direction m_{out}, since D is smooth. Choose φ such that $\varphi(m_{\text{out}}) = 1$ on all unbounded cells. Let $\mathscr{S}_\infty$ be the consistent wall structure defined by $(B, \mathscr{P}, \varphi)$. Then $\varphi(m_{\text{out}}) = 1$ along all walls of $\mathscr{S}_\infty$, since walls are not contained in the interior of the bounded maximal cell ([Grä], Lemma 5.14). Hence, each broken line for $\mathscr{S}_\infty$ has asymptotic monomial $m_1 = (qm_{\text{out}}, q)$ for some $q \in \mathbb{N}$.

Definition 2.3. For a point $P \in B_0$ and $q \in \mathbb{N}$ let $\mathfrak{B}_q(P)$ be the set of broken lines for the consistent wall structure $\mathscr{S}_\infty$ defined by $(B, \mathscr{P}, \varphi)$ with asymptotic monomial $m_1 = (qm_{\text{out}}, q)$ and endpoint $\mathfrak{b}(0) = P$.

Write $\mathfrak{B}_q^{(k)}(P)$ for the subset of $\mathfrak{B}_q(P)$ consisting of walls such that the ending monomial $a_\mathfrak{b}z^{m_\mathfrak{b}}$ has t-order $\leq k$. This means if $m_\mathfrak{b} = (\bar{m}, h)$, then $\varphi(-\bar{m}) \leq k - h$. Note that broken lines in $\mathfrak{B}_q^{(k)}(P)$ only break at walls of $\mathscr{S}_k$.

Proposition 2.4. *Let $\mathfrak{b}$ be a broken line in $\mathfrak{B}_q^{(k)}(P)$. If P lies in an unbounded chamber of $\mathscr{S}_k$, then $\bar{m}_\mathfrak{b}$ is parallel to m_{out}.*

Proof. Let P be a point in an unbounded chamber of $\mathscr{S}_k$. Suppose there exists a broken line $\mathfrak{b}$ in $\mathfrak{B}_m^{(k)}(P)$ such that $m_\mathfrak{b}$ is not parallel to m_{out}. Shift the point P in the direction of m_{out} until the last point at which $\mathfrak{b}$ breaks is a joint of $\mathscr{S}_k$. We do not cross any walls of $\mathscr{S}_k$ by doing this, since P lies in an unbounded chamber. Now by Proposition 2.2 the last segment of $\mathfrak{b}$ lies on a wall of $\mathscr{S}_k$, contradicting the assumption that P lies in an unbounded chamber of $\mathscr{S}_k$. □

Definition 2.5. Let $\mathfrak{B}_{p,q}(P)$ be the set of broken lines $\mathfrak{b}$ with endpoint $\mathfrak{b}(0) = P$, asymptotic direction $\bar{m}_1 = qm_{\text{out}}$, and ending direction $\bar{m}_{\mathfrak{b}} = -pm_{\text{out}}$.

Proposition 2.6. *For a broken line $\mathfrak{b}$ in $\mathfrak{B}_{p,q}(P)$ we have $m_1 = (qm_{\text{out}}, q)$ and $m_{\mathfrak{b}} = (-pm_{\text{out}}, q)$. In particular, the ending monomial $a_{\mathfrak{b}} z^{m_{\mathfrak{b}}}$ has t-order $p+q$.*

Proof. We have $m_1 = (qm_{\text{out}}, q)$ because z^{m_1} has t-order zero by the definition of broken lines. Since all walls in $\mathscr{S}_\infty$ are of the form $1 + a_{\mathfrak{p}} z^{(m_{\mathfrak{p}},0)}$, i.e., with zero as second exponent component, the ending monomial of $\mathfrak{b}$ is $a_{\mathfrak{b}} z^{m_{\mathfrak{b}}} = a_{\mathfrak{b}} z^{(-pm_{\text{out}},q)}$. Hence, the t-order of its ending monomial is $\varphi(pm_{\text{out}}) + q = p + q$. □

Corollary 2.7. *If P lies in an unbounded chamber of $\mathscr{S}_k$, then*

$$\mathfrak{B}_q^{(k)}(P) = \coprod_{p=1}^{k-q} \mathfrak{B}_{p,q}(P).$$

Proof. By Proposition 2.4 we have $\mathfrak{B}_q^{(k)}(P) \subset \coprod_{p=1}^{\infty} \mathfrak{B}_{p,q}(P)$. By Proposition 2.6 we have $\mathfrak{B}_{p,q}(P) \subset \mathfrak{B}_q^{(k)}(P)$ if and only if $p + q \leq k$. □

Corollary 2.8. *The set $\mathfrak{B}_{p,q}(P)$ is finite for all $p, q \in \mathbb{N}$.*

Proof. The set $\mathfrak{B}_q^{(k)}(P)$ is finite for all k, since broken lines in $\mathfrak{B}_q^{(k)}(P)$ only break at walls of $\mathscr{S}_k$ and $\mathscr{S}_k$ contains only finitely many walls. By Corollary 2.7 we have $\mathfrak{B}_{p,q}(P) \subset \mathfrak{B}_q^{(k)}(P)$ for $k \geq p + q$, so $\mathfrak{B}_{p,q}(P)$ is finite as well. □

3. Tropical curves

We use the definition of tropical curves from [Grä], Definition 3.3. Tropical curves may have bounded legs ending in affine singularities. They do not have uni- or bivalent vertices and fulfill the ordinary balancing condition $\sum_{E \ni V} u_{(V,E)} = 0$ at vertices V, if not specified otherwise.

Definition 3.1. A *tropical disk* $h : \Gamma \to B$ is a tropical curve with a unique univalent vertex V_∞, such that h is balanced for all vertices $V \neq V_\infty$.

Write $\mathfrak{H}_{p,q}(P)^\circ(P)$ for the set of all rational tropical disks on B with one unbounded leg of weight d and $h(V_\infty) = P$. Let $\mathfrak{H}_{p,q}(P)^\circ$ be the set of tropical disks in $\mathfrak{H}_q^\circ(P)$ with $u_{(V_\infty,E_\infty)} = -pm_{\text{out}}$, where E_∞ is the unique edge adjacent to V_∞.

Proposition 3.2. *There is a bijective map*

$$\mathfrak{B}_d(P) \to \mathfrak{H}_d^\circ(P), \mathfrak{b} \mapsto h_{\mathfrak{b}}^\circ.$$

Proof. It is easier to the define the map $\mathfrak{H}_d^\circ(P) \to \mathfrak{B}_d(P)$. A tropical disk in $\mathfrak{H}_d^\circ(P)$ is rational, i.e., has genus 0. Hence, its graph is a tree and there is a unique path from the unbounded edge to the vertex V_∞. This path defines a broken line in $\mathfrak{B}_d(P)$. The coefficients of its monomials are determined by $a_1 = 1$ and the breaking calculations.

The inverse map $\mathfrak{B}_d(P) \to \mathfrak{H}_d^\circ(P)$ is clear. Consider a broken line $\mathfrak{b}$ in $\mathfrak{B}_d(P)$. Whenever it breaks at a wall $\mathfrak{p}$, add a bounded edge/leg for $\mathfrak{p}$ and all its ancestors. By ancestors we mean all walls in $\mathscr{S}_\infty$ that are necessary to obtain $\mathfrak{p}$ via the scattering calculation. The weight of an edge E is the affine length of the exponent of the corresponding wall. □

Now let P be a general point in an unbounded chamber of $\mathscr{S}_d$.

Corollary 3.3. *We have a decomposition*

$$\mathfrak{H}_q^\circ(P) = \coprod_{p=1}^{\infty} \mathfrak{H}_{p,q}(P)^\circ.$$

In particular, for a tropical disk in $\mathfrak{H}_{p,q}(P)^\circ$ the edge E_∞ is parallel to m_{out}.

Proof. This follows from Corollary 2.7 and Proposition 3.2. □

Corollary 3.4. *There is a bijective map*

$$\mathfrak{B}_{p,q}(P) \to \mathfrak{H}_{p,q}^\circ(P), \mathfrak{b} \mapsto h_{\mathfrak{b}}^\circ.$$

Proof. The map from Proposition 3.2 maps broken lines in $\mathfrak{B}_{p,q}(P)$ to tropical disks in $\mathfrak{H}_{p,q}(P)^\circ(P)$. □

Definition 3.5. Let $\mathfrak{H}_{p,q}(P)$ be the set of tropical curves on B having two unbounded legs, of weight p and q, and such that the vertex of the unbounded leg of weight p is bivalent and mapped to P.

Proposition 3.6. *If P lies in an unbounded chamber of $\mathscr{S}_d$, then there is a bijective map*

$$\mathfrak{H}_{p,q}^\circ(P) \to \mathfrak{H}_{p,q}(P), h^\circ \mapsto h$$

Proof. By Proposition 2.4 and Corollary 3.4 tropical disks in $\mathfrak{H}_{p,q}^\circ(P)$ have ending edge E_∞ parallel to m_{out}. We obtain a tropical curve by completing E_∞ to an unbounded leg E_{out}. □

Corollary 3.7. *If P lies in an unbounded chamber of $\mathscr{S}_{p+q}$, then there is a bijective map*

$$\mathfrak{B}_{p,q}(P) \to \mathfrak{H}_{p,q}(P), \mathfrak{b} \mapsto h_{\mathfrak{b}}$$

given by the composition of the maps from Propositions 3.2 and 3.6.

Corollary 3.8. *The set $\mathfrak{H}_{p,q}(P)$ is finite.*

4. Tropical correspondence

In [Grä] the author established a correspondence between wall functions and 1-marked log Gromov-Witten invariants. Here we prove a similar correspondence between broken lines and 2-marked log Gromov-Witten invariants. The steps of the proof are the same, corresponding to the subsections of this section:

(1) resolve the log singularities to obtain a log smooth degeneration;
(2) refine $\mathscr{P}$ by all walls and also by the broken line (that's new here) to obtain toric transversality;
(3) apply the degeneration formula and show that there is only one possible way of gluing;
(4) (4) inductively apply the correspondence of [GPS] between log Gromov-Witten invariants and walls for toric varieties, and use that broken line calculations are similar to scattering calculations (Proposition 2.2).

Steps (1)-(3) will lead to a definition of the multiplicity $\text{Mult}(h)$ of a tropical curve h, only depending on known data, such that

$$N_{p,q} = p \cdot \sum_{h \in \mathfrak{H}_{p,q}(P)} \text{Mult}(h).$$

Step (4) will show that $a_{\mathfrak{b}} = \text{Mult}(h_{\mathfrak{b}})$.

4.1. **Resolution of singularities.** To apply the degeneration formula we need a log smooth family. Unfortunately, our toric degeneration is not even coherent (there is no chart for the log structure at the points corresponding to the affine singularities). However, we can obtain a log smooth degeneration without changing the general fiber by successively blowing up $\mathfrak{X}$ along the irreducible components of the central fiber. Depending on the number of components there are several ways to do this. The easiest is to choose a cyclic ordering (with respect to the intersection combinatorics) and to blow up in this order along all but one irreducible component. Figure 4.1 shows the resulting central fiber (intersection complex) for $\mathbb{F}_1$. The resulting family $\tilde{\mathfrak{X}} \rightarrow \mathbb{A}^1$ has general fiber X and is log smooth by [GS2], Lemma 2.12. In [Grä], §2, the author used a more symmetric resolution. However, projectivity of this resolution is not clear, so we don't use it here.

For later convenience we indicate the choices of small resolutions by red stubs attached to the vertices of $\mathscr{P}$. The stubs at a vertex v point in the directions corresponding to the toric divisors of X_v intersecting an exceptional line. We denote the primitive vectors in the direction of the red stubs adjacent to v by $m_{v,+,i} \in \Lambda_{\tilde{B},v}$ for $i = 1,\ldots,n$, where $n \in \{0,1,2\}$ is the number of exceptional lines contained in X_v. Denote the primitive vectors in the direction of the other edges of σ_0 adjacent to v by $m_{v,-,i} \in \Lambda_{\tilde{B},v}$ for $i = 1, \ldots, n$, where $m = 2 - n$. Further, m_{out} is the unique unbounded direction of $\tilde{B}$.

Example 4.1. Figure 4.1 shows the intersection complex and its dual for a resolution of the toric degeneration of $(\mathbb{P}^2, E)$. One component (corresponding to σ and v, respectively) contains two exceptional lines.

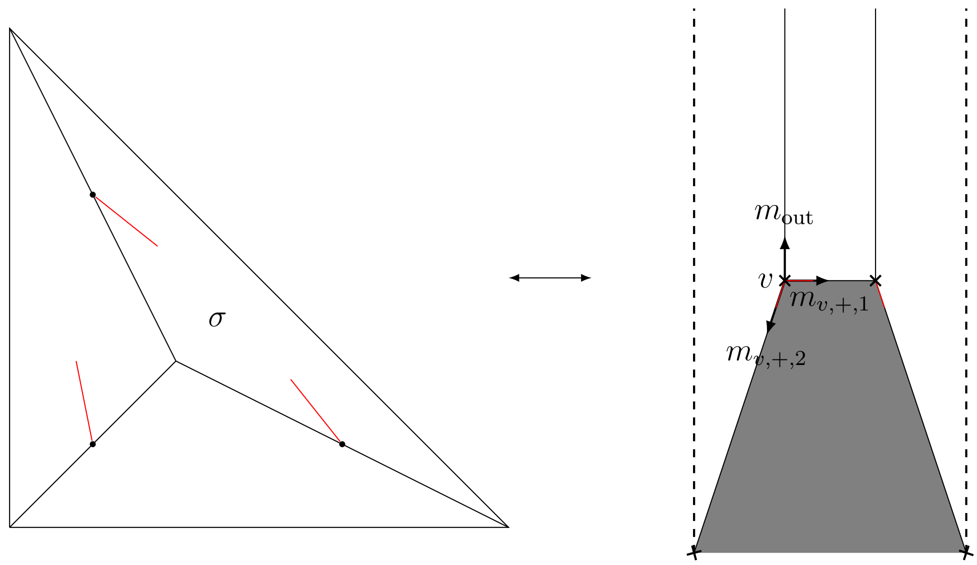


FIGURE 4.1. The intersection complex and its dual $(\tilde{B}, \mathscr{P}, \varphi)$ for a resolution of the toric degeneration of $(\mathbb{P}^2, E)$.

Definition 4.2. For an effective curve class β of X and $p, q \in \mathbb{N}$ with $p+q = D \cdot \beta$ define a class of stable log maps $\beta_{p,q}$ to $\tilde{\mathfrak{X}} \to \mathbb{A}^1$ as follows:
(1) genus $g = 0$;
(2) fibers have curve class β;
(3) 2 marked points x_p, x_q having contact orders p, q with $\mathfrak{D}$.

Let $\mathscr{M}(\tilde{\mathfrak{X}}/\mathbb{A}^1, \beta_{p,q})$ be the moduli space of stable log maps to $\tilde{\mathfrak{X}} \to \mathbb{A}^1$ of class $\beta_{p,q}$. By [GS5] this is a proper Deligne-Mumford stack and admits a virtual fundamental class $[\![\mathscr{M}(\mathfrak{X}/\mathbb{A}^1, \beta_{p,q})]\!]$. It has virtual dimension 1, since the contact orders cut down the virtual dimension by $(p-1)+(q-1) = D \cdot \beta - 2$. Let $\text{ev} : \mathscr{M}(\tilde{\mathfrak{X}}/\mathbb{A}^1, \beta_{p,q}) \to \mathfrak{D}$ be the evaluation map at x_p.

Definition 4.3. Define the 2-marked log Gromov-Witten invariant

$$N_{p,q}(X, \beta) = \int_{[\![\mathscr{M}(\mathfrak{X}/\mathbb{A}^1, \beta_{p,q})]\!]} \text{ev}^\star[\text{pt}] \in \mathbb{Q}.$$

Since log Gromov-Witten invariants are constant in log smooth families ([MR], Appendix A), this agrees with the definition of $N_{p,q}(X, \beta)$ in the introduction.

Remark 4.4. We believe that there should be a definition of log Gromov-Witten invariants for relatively coherent targets that captures all information about the resolution described here. This would simplify the subsequent proof a lot as we don't have to translate between the toric degeneration and its resolution all the time. However, this is not yet worked out.

4.2. **Tropical curves and refinement.** It turns out that tropical curves on $\tilde{B}$ that are tropicalizations of stable log maps to $\tilde{X}$ do not fulfill the ordinary balancing condition, but the following modified one.

Definition 4.5. For $p,q \in \mathbb{Z}_{>0}$ and P a general point in a unbounded chamber of $\mathscr{S}_{p+q}$ let $\tilde{\mathfrak{H}}_{p,q}(P)$ be the set of tropical curves on $\tilde{B}$ with two unbounded legs, of weights p and q, with vertex of the unbounded leg of weight p being bivalent and mapping to P, and such that each vertex is of one of the following types:

(I) V is not mapped to a vertex of $\mathscr{P}$. Then the ordinary balancing condition holds:
$$\sum_{E\ni V} u_{(V,E)} = 0.$$
The sum is over all edges or legs $E \in E(\Gamma_C) \cup L(\Gamma_C)$ containing V.

(II) V is mapped to a vertex v of $\mathscr{P}$, and is 1-valent with adjacent edge E mapped onto the edge of $\mathscr{P}$ containing the red stub adjacent to v. Then the balancing condition reads, with $m_{v,+,i}$ as in Figure 4.1 for some i,
$$u_{(V,E)} = km_{v,+,i}.$$

(III) V is mapped to a vertex v of $\mathscr{P}$ and has exactly one adjacent edge or leg $E_{V,\text{out}}$ that is not mapped onto a compact edge of $\mathscr{P}$. All other edges (possibly none) are compact with other vertex of type (II) above. In this case, for some $k_i \geq 0$, the following balancing condition holds:
$$\sum_{E\ni V} u_{(V,E)} + \sum_{i=1}^{n(v)} k_i m_{v,+,i} = 0.$$

Write $V_{(I)}(\tilde{\Gamma})$, $V_{(II)}(\tilde{\Gamma})$, $V_{(III)}(\tilde{\Gamma})$ for the set of vertices of type (I), (II), (III).

The *class* of a tropical curve is the class of the corresponding stable log map. It can be read off from the intersection with unbounded edges of $\mathscr{P}$ or, equivalently, via projection to the unbounded direction as in [Grä], §3.4.

Definition 4.6. For $p,q \in \mathbb{Z}_{>0}$, a general point P in an unbounded chamber of $\mathscr{S}_{p+q}$ and an effective curve class β on X, let $\tilde{\mathfrak{H}}^\beta_{p,q}(P)$ be the moduli space of tropical curves in $\tilde{\mathfrak{H}}_{p,q}(P)$ of class β. Note that
$$\tilde{\mathfrak{H}}_{p,q}(P) = \coprod_\beta \tilde{\mathfrak{H}}^\beta_{p,q}(P),$$
where the sum is over all effective curve classes on X with $D \cdot \beta = p + q$.

Proposition 4.7. *$\tilde{\mathfrak{H}}^{\beta}_{p,q}(P)$ is the set of tropical curves that arise as tropicalizations of stable log maps in $\mathscr{M}(X,\beta_{p,q})$. Vertices of type (II) correspond to components that are multiple covers of an exceptional $\mathbb{P}^1$, and vertices of type (III) to components intersecting an exceptional $\mathbb{P}^1$.*

Proof. It is shown in [Grä], Proposition 3.12, that tropicalizations of stable log maps to $\tilde{\mathfrak{X}}$ have vertices of types (I)-(III) above. The condition to have two marked points with contact orders p and q is equivalent to the condition on the tropical curve to have two unbounded legs of weight p and q, respectively. □

Lemma 4.8. *There is a surjective map $\mathfrak{H}_{p,q}(P) \to \tilde{\mathfrak{H}}_{p,q}(P)$ by deleting bounded legs in the directions $m_{v,+,i}$ (the directions of the red stubs) and extending bounded legs in the directions $m_{v,-,i}$. See Figure 4.2 for an example and* [Grä]*, Construction 3.17, for details of the construction.*

Corollary 4.9. *The set $\tilde{\mathfrak{H}}_{p,q}(P)$ is finite.*

Proof. This follows from Corollary 3.8 and Lemma 4.8. □

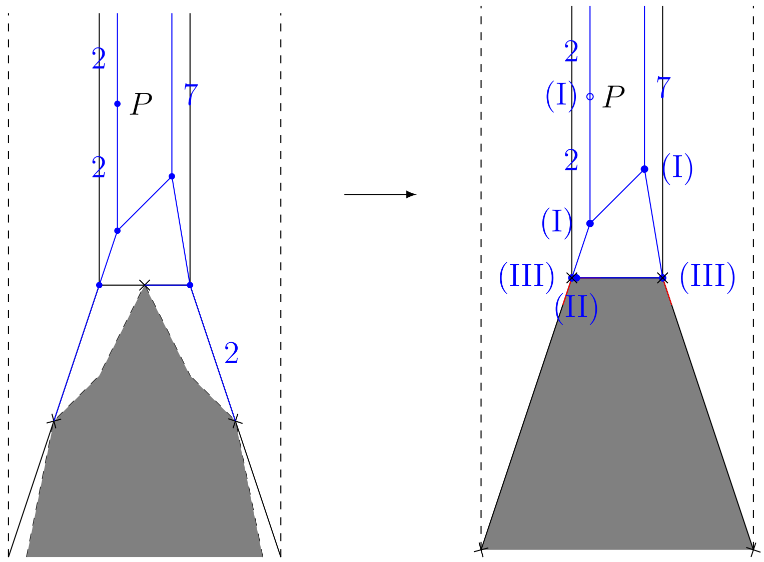


FIGURE 4.2. An example of the map $\mathfrak{H}_{2,7} \to \tilde{\mathfrak{H}}_{2,7}$. The weights of edges and the types (I)-(III) of vertices of the target are given. Two vertices are mapped to the same vertex, but not connected.

For $p,q \in \mathbb{Z}_{>0}$ let P be a general point in an unbounded chamber of $\mathscr{S}_{p+q}$. Let $\mathscr{P}_{p,q}$ be a refinement of $\mathscr{P}$ such that all tropical curves in $\mathfrak{H}_{p,q}(P)$ are contained in the 1-skeleton of $\mathscr{P}_{p,q}$ and P is a vertex of $\mathscr{P}_{p,q}$. This induces a logarithmic modification $\tilde{\mathfrak{X}}_{p,q} \to \mathbb{A}^1$ of $\tilde{\mathfrak{X}} \to \mathbb{A}^1$ via subdivision of Artin fans, see [AW]. By making a base change $t \mapsto t^e$ we can scale $\mathscr{P}_{p,q}$ and thus assume it has integral vertices. By construction, all stable log maps to the central fiber Y of $\tilde{\mathfrak{X}}_{p,q} \to \mathbb{A}^1$ are torically transverse.

4.3. **The degeneration formula.** Gromov-Witten invariants are invariant under logarithmic modifications [AW]. Hence we can compute $N_{p,q}(X,\beta)$ on Y:

$$N_{p,q}(X,\beta) = \int_{[\![\mathscr{M}(Y,\beta_{p,q})]\!]} \mathrm{ev}^\star[\mathrm{pt}],$$

where ev is the evaluation map at x_p, the marked point of order p.

On the central fiber Y we have some techniques for computing the invariants $N_{p,q}(X,\beta)$. By the decomposition formula (Proposition 4.10) the connected components of the moduli space $\mathscr{M}(Y,\beta_{p,q})$ are labelled by certain (decorated) tropical curves. The gluing formula (Proposition 4.15) relates the contributions of every tropical curve to contributions of its vertices and edges. In our case, similar to [Grä], the situation is particularly easy. Tropical curves in $\tilde{\mathfrak{H}}_{p,q}(P)$ have a natural orientation of edges and the gluing according to this orientation is the only one giving a nonzero contribution (Proposition 4.16).

Proposition 4.10 (Decomposition formula)**.** *For $\tilde{h} \in \tilde{\mathfrak{H}}^\beta_{p,q}(P)$ let $\mathscr{M}_{\tilde{h}}$ be the moduli space of stable log maps in $\mathscr{M}(Y,\beta_{p,q})$ with tropicalization $\tilde{h}$. Then*

$$[\![\mathscr{M}(Y,\beta_{p,q})]\!] = \sum_{\tilde{h}\in\tilde{\mathfrak{H}}^\beta_{p,q}(P)} \frac{l_{\tilde{\Gamma}}}{|\mathrm{Aut}(\tilde{h})|} F_\star[\![\mathscr{M}_{\tilde{h}}]\!],$$

where $l_{\tilde{\Gamma}} := \mathrm{lcm}\{w_E \mid E \in E(\tilde{\Gamma})\}$ and $F : \mathscr{M}_{\tilde{h}} \to \mathscr{M}_\beta$ is the forgetful map.

Proof. This is a special case of the decomposition formula of [ACGS1], similar to [Grä], Proposition 4.4. In [ACGS1] the sum is over tropical curves with vertices decorated by curve classes to the corresponding components. In our case, as in [Grä], Proposition 4.1, these curve classes are determined by the tropical curve. Hence, we can simply sum over $\tilde{\mathfrak{H}}_{p,q}(P)$. The nominator $l_{\tilde{\Gamma}}$ is the smallest integer such that scaling $\tilde{B}$ by $l_{\tilde{\Gamma}}$ leads to a tropical curve with integral vertices and edge lengths. By construction $\mathscr{P}_{p,q}$ has integral vertices and tropical curves in $\tilde{\mathfrak{H}}_{p,q}(P)$ are contained in the 1-skeleton of $\mathscr{P}_{p,q}$ with vertices mapping to vertices of $\mathscr{P}_{p,q}$. The affine length of the image of an edge E is a multiple of w_E. So the scaling necessary to obtain integral edge lengths is $l_{\tilde{\Gamma}} = \mathrm{lcm}\{w_E \mid E \in E(\tilde{\Gamma})\}$. □

Let $\tilde{h} : \tilde{\Gamma} \to \tilde{B}$ be a tropical curve in $\tilde{\mathfrak{H}}_{p,q}(P)$. For a vertex V of $\tilde{\Gamma}$ define

$$\mathscr{M}_V^\circ := \mathscr{M}(Y^\circ_{\tilde{h}(V)}, i_V^\star\beta_{p,q}),$$

where $Y^\circ_{\tilde{h}(V)}$ is the complement of the 0-dimensional toric strata in $Y_{\tilde{h}(V)}$ and $i_V : Y^\circ_{\tilde{h}(V)} \to Y$ is the inclusion map.

For $V \in V_{II}(\tilde{\Gamma})$ (Definition 4.5) with adjacent edge E, the moduli space $\mathscr{M}_V^\circ$ is proper, since it is isomorphic to the moduli space of w_E-fold multiple covers of $\mathbb{P}^1$ totally ramified at a point. For $V \in V(\tilde{\Gamma}) \setminus V_{II}(\tilde{\Gamma})$ we obtain a proper moduli space as follows. Since tropical curves in $\tilde{\mathfrak{H}}_{p,q}(P)$ have genus 0, the graph $\tilde{\Gamma}$ is a tree. We give it the structure of a rooted tree by choosing the vertex V_{out} of the unbounded leg of weight q to be the root vertex. Then there is a natural orientation of the edges of $\tilde{\Gamma}$ by choosing edges to point from a vertex to its parent. For each vertex $V \in V(\tilde{\Gamma}) \setminus V_{II}(\tilde{\Gamma})$ there is an evaluation map

$$\mathrm{ev}^{\circ}_{V,-} : \mathscr{M}^{\circ}_V \to \prod_{E \to V} D^{\circ}_E,$$

where the product is over all edges of $\tilde{\Gamma}$ adjacent to V and pointing towards V.

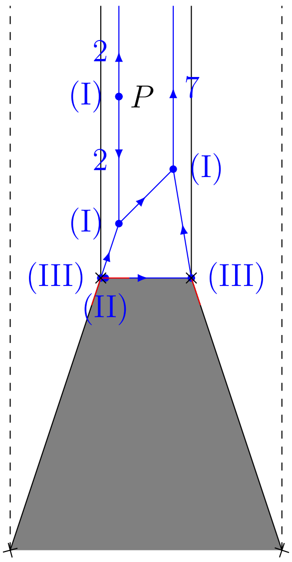


FIGURE 4.3. The orientation of the tropical curve from Figure 4.2.

Lemma 4.11. *The evaluation map* $\mathrm{ev}^{\circ}_{V,-}$ *is proper.*

Proof. Combine [GPS], Propositions 4.2 and 5.1, as in [Grä], Lemma 4.5. □

Since properness of morphisms is stable under base change, we obtain a proper moduli space by base change to a point $\gamma_V : \mathrm{Spec}\ \mathbb{C} \to \prod_{E \to V} D^{\circ}_E$, that is,

$$\mathscr{M}_{\gamma_V} := \mathrm{Spec}\ \mathbb{C} \times_{\prod_{E \to V} D^{\circ}_E} \mathscr{M}^{\circ}_V$$

is a proper Deligne-Mumford stack.

Lemma 4.12. *For* $V \in V_{II}(\tilde{\Gamma})$ *the virtual dimension of* $\mathscr{M}_V$ *is zero. Otherwise the virtual dimension of* $\mathscr{M}_V$ *equals the codimension of* γ_V*.*

Proof. See [GPS], §5.3, and [Grä], Lemma 4.6. □

Definition 4.13. For a vertex V of $\tilde{\Gamma}$ define

$$N_V := \begin{cases} \int_{[\![\mathscr{M}^{\circ}_V]\!]} 1, & V \in V_{II}(\tilde{\Gamma}); \\ \int_{\mathscr{M}_{\gamma_V}} \gamma_V^! [\![\mathscr{M}^{\circ}_V]\!], & V \in V_I(\tilde{\Gamma}) \cup V_{III}(\tilde{\Gamma}). \end{cases}$$

This is a finite number by Lemma 4.12 and independent of γ_V by Lemma 4.11.

Proposition 4.14 ([Grä], Proposition 4.8)**.**

(I) For $V \in V_I(\tilde{\Gamma})$ let $e_1,\ldots,e_n$ be the edges of $\mathscr{P}_{p,q}$ adjacent to $\tilde{h}(V)$ and let $m_1,\ldots,m_n$ be the corresponding primitive vectors. Let $\boldsymbol{w}_i = (w_{i1},\ldots,w_{il_i})$ be the weights of edges of $\tilde{\Gamma}$ mapping to e_i and write $\boldsymbol{w} = (\boldsymbol{w}_1,\ldots,\boldsymbol{w}_n)$. Then N_V is the toric invariant $N_{\boldsymbol{m}}(\boldsymbol{w})$ as defined in [GPS] *and* [Grä], *§4.1.*

(II) If $V \in V_{II}(\tilde{\Gamma})$, then

$$N_V = \frac{(-1)^{w_E-1}}{w_E^2},$$

where E is the unique edge adjacent to V.

(III) If $V \in V_{III}(\tilde{\Gamma})$, then

$$N_V = \sum_{i=1}^{n(v)} \sum_{\boldsymbol{w}_{V,+}} \frac{N_{\boldsymbol{m}}(\boldsymbol{w})}{|\mathrm{Aut}(\boldsymbol{w}_{V,+})|} \prod_{i=1}^{l} \frac{(-1)^{w_{V,i}-1}}{w_{V,i}}.$$

Here $n(v)$ is the number of red stubs attached to v (see §4.1). The second sum is over all weight vectors $\boldsymbol{w}_{V,+,i} = (w_1,\ldots,w_{l_i})$ such that $\sum_{i=1}^{l_i} w_i = k_i$, with k_i as in Proposition 4.7, (III). Further, $N_{\boldsymbol{m}}(\boldsymbol{w})$ is a toric invariant as in (I) with $\boldsymbol{m} = ((m_{v,-,i})_i,(m_{v,+,i})_i)$ and $\boldsymbol{w} = (((w_E)_{E\in E_{V,-,i}})_i,(\boldsymbol{w}_{V,+,i})_i)$, where $E_{V,-,i}$ is the set of edges adjacent to V and mapped to direction $m_{v,-,i}$.

The bivalent vertex V mapping to P is of type (I) and contributes $N_V = 1$.

Define $\bigtimes_{V\in V(\tilde{\Gamma})} \mathscr{M}_V$ to be the moduli space of stable log maps in $\prod_V \mathscr{M}_V$ matching over the divisors D_E, $E \in E(\tilde{\Gamma})$, i.e., the fiber product

$$\begin{array}{ccc} \bigtimes_V \mathscr{M}_V & \longrightarrow & \prod_V \mathscr{M}_V \\ \downarrow & & \downarrow \mathrm{ev} \\ \prod_{E\in E(\tilde{\Gamma})} D_E & \xrightarrow{\ \delta\ } & \prod_V \prod_{\substack{E\in E(\tilde{\Gamma})\\ V\in E}} D_E \end{array}$$

By [KLR] there is an étale morphism $\mathrm{cut} : \mathscr{M}_{\tilde{h}} \to \bigtimes_V \mathscr{M}_V^\circ$ of degree $\deg(\mathrm{cut}) = (\prod_{E\in E(\tilde{\Gamma})} w_E)/l_{\tilde{\Gamma}}$, where $l_{\tilde{\Gamma}} = \mathrm{lcm}\{w_E\}$. By compatibility of obstruction theories ([KLR], §9) we have

$$[\![\mathscr{M}_{\tilde{h}}]\!] = \mathrm{cut}^\star \delta^! \prod_{V\in V(\tilde{\Gamma})} [\![\mathscr{M}_V^\circ]\!].$$

By the projection formula $\mathrm{cut}_\star\mathrm{cut}^\star$ is multiplication with $\deg(\mathrm{cut})$, so

$$\mathrm{cut}_\star[\![\mathscr{M}_{\tilde{h}}]\!] = \frac{1}{\ell_{\tilde{\Gamma}}} \prod_{E\in E(\tilde{\Gamma})} w_E \cdot \delta^! \prod_{V\in V(\tilde{\Gamma})} [\![\mathscr{M}_V^\circ]\!].$$

Proposition 4.15 (Gluing formula)**.**

$$\int_{[\![\mathscr{M}_{\tilde{h}}]\!]} 1 = \frac{1}{\ell_{\tilde{\Gamma}}} \prod_{E\in E(\tilde{\Gamma})} w_E \cdot \int_{\delta^! \prod_V [\![\mathscr{M}_V]\!]} 1.$$

Proof. By the above formula, the cycles $\mathrm{cut}_\star[\![\mathscr{M}_{\tilde{h}}]\!]$ and $\frac{1}{\ell_{\tilde{\Gamma}}}\prod_{E\in E(\tilde{\Gamma})} w_E\cdot\delta^!\prod_V[\![\mathscr{M}_V]\!]$ have the same restriction to the open substack $\bigtimes_V \mathscr{M}_V^\circ$ of $\bigtimes_V \mathscr{M}_V$. Hence by [Ful], Proposition 1.8, their difference is rationally equivalent to a cycle supported on the closed substack $Z := (\bigtimes_V \mathscr{M}_V)\setminus(\bigtimes_V \mathscr{M}_V^\circ)$. Suppose there exists an element $(f_V : C_V \to Y_{\tilde{h}(V)})_{V\in V(\tilde{\Gamma})}\in Z$. Then at least one of the source curves C_V would contain a nontrivial cycle of components as can be seen by a loop construction as in the proof of [GPS], Proposition 4.2, or [Grä], Lemma 4.5. This contradicts $g=0$, so Z is empty, completing the proof. □

Proposition 4.16 (Unique gluing)**.**

$$\int_{\delta^!\prod_V[\![\mathscr{M}_V]\!]} 1 = \prod_{V\in V(\tilde{\Gamma})} N_V.$$

Proof. This is similar to the proof of [Grä], Proposition 4.13. As before, $\tilde{\Gamma}$ is a rooted tree with root vertex V_{out}. This gives an orientation of the edges of $\tilde{\Gamma}$. The only gluing that gives a nonzero contribution after integration is the one according to this orientation. Any other gluing gives a negative virtual dimension, since we have too many conditions on one of the irreducible components. □

Theorem 4.17 (Degeneration formula)**.** *Let P be a point in an unbounded chamber of $\mathscr{S}_{p+q}$. Then*

$$N_{p,q}(X,\beta) = \sum_{\tilde{h}\in\tilde{\mathfrak{H}}^\beta_{p,q}(P)} \frac{1}{|\mathrm{Aut}(\tilde{h})|}\cdot\prod_{E\in E(\tilde{\Gamma})} w_E\cdot\prod_{V\in V(\tilde{\Gamma})} N_V.$$

Proof. Since the virtual dimension of $\mathscr{M}_\beta$ is zero, integration (i.e., proper push-forward to a point) of the decomposition formula (Proposition 4.10) gives

$$N_{p,q}(X,\beta) = \sum_{\tilde{h}\in\tilde{\mathfrak{H}}^\beta_{p,q}(P)} \frac{1}{|\mathrm{Aut}(\tilde{h})|}\int_{[\![\mathscr{M}_{\tilde{h}}]\!]} 1.$$

Using Propositions 4.15 and 4.16 we get the above formula. □

We obtain a more symmetric version of this formula by summing over balanced tropical curves in $\mathfrak{H}^\beta_{p,q}(P)$.

Definition 4.18. Let $h:\Gamma\to B$ be a tropical curve in $\mathfrak{H}_\beta$ and let V be a vertex of Γ. Then the image of V under the map from Lemma 4.8 is a vertex of $\tilde{\Gamma}$ of type (I) or (III). Let $\mathbf{m}$ and $\mathbf{w}$ be as in the respective case of Proposition 4.14 and define $N_V^{\mathrm{tor}} := N_{\mathbf{m}}(\mathbf{w})$. Note that $N_V^{\mathrm{tor}} = N_V$ for vertices of type (I).

Definition 4.19. Define $N^{\mathrm{trop}}_{p,q}(X,\beta) = \sum_{h\in\mathfrak{H}^\beta_{p,q}(P)} N_h$, where P is a point in an unbounded chamber of $\mathscr{S}_{p+q}$ and

$$N_h := \frac{1}{|\mathrm{Aut}(h)|}\cdot\prod_{E\in E(\Gamma)} w_E\cdot\prod_{E\in L_\Delta(\Gamma)}\frac{(-1)^{w_E-1}}{w_E}\cdot\prod_{V\in V(\Gamma)} N_V^{\mathrm{tor}}.$$

Here $L_\Delta(\Gamma)$ is the set of bounded legs of Γ.

Definition 4.20. Let $h:\Gamma\rightarrow B$ be a tropical curve. For a trivalent vertex $V\in V(\Gamma)$ define $m_V=|u_{(V,E_1)}\wedge u_{(V,E_2)}|=|\det(u_{(V,E_1)}|u_{(V,E_2)})|$, where E_1,E_2 are any two edges adjacent to V. For a vertex $V\in V(\Gamma)$ of valency $\nu_V>3$ let h_V be the one-vertex tropical curve describing h locally at V and let h_V' be a deformation of h_V to a trivalent tropical curve. It has ν_V-2 vertices. Define $m_V=\prod_{V'\in V(h_V')}m_{V'}$. For a bounded leg $E\in L_\Delta(\Gamma)$ define $m_E=(-1)^{w_E+1}/w_E^2$. Then define the *multiplicity* of h to be

$$\text{Mult}(h)=\frac{1}{|\text{Aut}(h)|}\cdot\prod_V m_V\cdot\prod_{E\in L_\Delta(\Gamma)}m_E.$$

Proposition 4.21. *For a tropical curve $h:\Gamma\rightarrow B$ in $\mathfrak{H}_d$ we have*

$$N_h=\text{Mult}(h)$$

Proof. By the tropical correspondence theorem with point conditions on toric divisors ([GPS], Theorem 3.4) we have $m_V=\prod_{E\rightarrow V}w_E\cdot N_V^{\text{tor}}$. The product is over all eges of Γ pointing towards V with respect to the orientation of Γ such that all edges point towards the root vertex V_{out}. Then $\prod_{V\in V(\Gamma)}m_V=\prod_{E\in E(\Gamma)}w_E\cdot\prod_{V\in V(\Gamma)}N_V^{\text{tor}}$ as each $E\in E(\Gamma)$ occurs exactly once. Plugging this and $m_E=(-1)^{w_E+1}/w_E^2$ into the definition of Mult(h) we obtain N_h. □

Theorem 4.22 (Tropical correspondence theorem)**.**

$$N_{p,q}(X,\beta)=pN_{p,q}^{\text{trop}}(X,\beta).$$

Proof. This is a simple rearrangement of equations as in [Grä], Theorem 4.17. □

4.4. **Broken line calculations.**

Definition 4.23. Let $\mathfrak{p}\in\mathscr{S}_\infty$ be a wall and choose $x\in\text{Int}(\mathfrak{p})$. Define $\mathfrak{H}_{\mathfrak{p},d}^\circ$ to be the set of all tropical disks $h:\Gamma\rightarrow B$ with $h(V_\infty)=x$ and $u_{(V_\infty,E_\infty)}=-d\cdot m_{\mathfrak{p}}$. Note that the sets $\mathfrak{H}_{\mathfrak{p},d}$ are in bijection for different choices of $x\in\text{Int}(\mathfrak{p})$. For $h\in\mathfrak{H}_{\mathfrak{p},w}^\circ$ define Mult(h) as in Definition 4.19.

Proposition 4.24 ([Grä], Proposition 5.20)**.** *For a wall $\mathfrak{p}$ of $\mathscr{S}_\infty$ we have*

$$\log f_{\mathfrak{p}}=\sum_{w=1}^\infty\sum_{h\in\mathfrak{H}_{\mathfrak{p},w}}\text{Mult}(h)z^{(wm_{\mathfrak{p}},0)}.$$

Theorem 4.25.

$$a_{\mathfrak{b}}=\text{Mult}(h_{\mathfrak{b}}).$$

Proof. This follows from Propositions 2.2 and 4.24. Since breaking is similar to scattering (Proposition 2.2), at each breaking point $x\in\text{Int}(\mathfrak{p})$ of $\mathfrak{b}$ we pick up a factor of Mult(h_x) for $h_x\in\mathfrak{H}_{\mathfrak{p},d}^\circ$ a tropical sub-disk of $h_{\mathfrak{b}}$. Together these factors exactly give Mult$(h_{\mathfrak{b}})$. □

5. Theta functions

Definition 5.1. For a point $P \in B_0$ and an asymptotic direction m define the corresponding *theta function* by

$$\vartheta_m(P) = \sum_{\mathfrak{b} \in \mathfrak{B}_m(P)} a_{\mathfrak{b}} z^{m_{\mathfrak{b}}}$$

In our case, with smooth divisor D, the dual intersection complex B has exactly one unbounded direction m_{out}, so asymptotic directions on B are just multiples of m_{out}. We write $\vartheta_d(P)$ for $\vartheta_{d \cdot m_{\mathrm{out}}}(P)$ with $d \in \mathbb{N}$.

Proposition 5.2 ([GHS], Theorem 3.24, [GS6], Theorem 1.9)**.** *Theta functions generate a commutative ring (associative if X is Fano) with unit ϑ_0 by the multiplication rule*

$$\vartheta_p(P) \cdot \vartheta_q(P) = \sum_{r=0}^{\infty} \alpha_{p,q}^r(P) \vartheta_r(P)$$

with structure constants

$$\alpha_{p,q}^r(P) = \sum_{\substack{(\mathfrak{b}_1,\mathfrak{b}_2) \in \mathfrak{B}_p(P) \times \mathfrak{B}_q(P) \\ m_{\mathfrak{b}_1} + m_{\mathfrak{b}_2} = r}} a_{\mathfrak{b}_1} a_{\mathfrak{b}_2}$$

Theorem 5.3 (Theorems 1 and 2)**.** *Write $x = z^{(-m_{\mathrm{out}},-1)}$ and $t = x^{(0,1)}$. Then*

$$\vartheta_q = z^{-q} + \sum_{p=1}^{\infty} N_{p,q}^{\mathrm{trop}} t^{p+q} x^p \tag{1}$$

Moreover, $\alpha_{p,q}^r = 1$ if $r = p+q$ and otherwise

$$\alpha_{p,q}^r = (N_{p-r,q}^{\mathrm{trop}} + N_{q-r,p}^{\mathrm{trop}}) t^{p+q-r}, \tag{2}$$

where we define $N_{p,q}^{\mathrm{trop}} = 0$ whenever $p \leq 0$.

Proof. This follows from 4.25. ☐

Theorems 4.22 and 5.3 together give Theorems 1 and 2. Plugging both expressions of Theorem 5.3 into the multiplication rule we obtain relations among the $N_{p,q}$. Since powers of ϑ_1 generate the theta ring these equations determine all $N_{p,q}$ by only knowing the invariants $N_{p,1}$ or, equivalently, the invariants $N_{1,q}$.

Example 5.4. We use Theorem 5.3 to obtain relations among the 2-marked invariants $N_{p,q}$ for $\mathbb{P}^2$ up to order $d = (p+q)/3 = 2$. By (3) we have

$$\begin{aligned}
\vartheta_1 &= x^{-1} + 2N_{2,1}t^3x^2 + 5N_{5,1}t^6x^5 + \mathcal{O}(t^9) \\
\vartheta_2 &= x^{-2} + N_{1,2}t^3x + 4N_{4,2}t^6x^4 + \mathcal{O}(t^9) \\
\vartheta_3 &= x^{-3} + 3N_{3,3}t^6x^3 + \mathcal{O}(t^9) \\
\vartheta_4 &= x^{-4} + 2N_{2,4}t^6x^2 + \mathcal{O}(t^9) \\
\vartheta_5 &= x^{-5} + N_{1,5}t^6x + \mathcal{O}(t^9)
\end{aligned}$$

By direct mutliplication we get

$$\vartheta_1 \cdot \vartheta_1 = x^{-2} + 4N_{2,1}t^3x + (4N_{2,1}^2 + 10N_{5,1})t^6x^4 + \mathcal{O}(t^9).$$

On the other hand, (4) gives, with $\alpha_{1,1}^1 = 0$,

$$\vartheta_1 \cdot \vartheta_1 = \vartheta_2 = x^{-2} + N_{1,2}t^3x + 4N_{4,2}t^6x^4 + \mathcal{O}(t^9)$$

Comparing these two equations we get the relations

$$N_{1,2} = 4N_{2,1}, \qquad 2N_{4,2} = 2N_{2,1}^2 + 5N_{5,1}$$

Similarly, comparing

$$\vartheta_1 \cdot \vartheta_2 = x^{-3} + (N_{1,2} + 2N_{2,1})t^3x^0 + (2N_{1,2}N_{2,1} + 4N_{4,2} + 5N_{5,1})t^6x^3 + \mathcal{O}(t^9)$$

with

$$\vartheta_1 \cdot \vartheta_2 = \vartheta_3 + (N_{1,2} + 2N_{2,1})t^3\vartheta_0 = x^{-3} + (N_{1,2} + 2N_{2,1})x^0 + 3N_{3,3}x^3 + \mathcal{O}(t^9)$$

we get

$$3N_{3,3} = 2N_{1,2}N_{2,1} + 4N_{4,2} + 5N_{5,1}.$$

Comparing

$$\vartheta_1 \cdot \vartheta_3 = x^{-4} + 2N_{2,1}t^3x^{-1} + (3N_{3,3} + 5N_{5,1})t^6x^2 + \mathcal{O}(t^9)$$

with

$$\vartheta_1 \cdot \vartheta_3 = \vartheta_4 + 2N_{2,1}t^3\vartheta_1 = x^{-4} + 2N_{2,1}t^3x^{-1} + (4N_{2,1}^2 + 2N_{2,4})t^6x^2 + \mathcal{O}(t^9)$$

we get the relation

$$3N_{3,3} + 5N_{5,1} = 4N_{2,1}^2 + 2N_{2,4}$$

and comparing

$$\vartheta_1 \cdot \vartheta_4 = x^{-5} + 2N_{2,1}t^3x^{-2} + (2N_{2,4} + 5N_{5,1})t^6x + \mathcal{O}(t^9)$$

with

$$\vartheta_1 \cdot \vartheta_4 = \vartheta_5 + 2N_{2,1}t^3\vartheta_2 = x^{-5} + 2N_{2,1}t^3x^{-2} + (N_{1,5} + 2N_{1,2}N_{2,1})t^6x + \mathcal{O}(t^9)$$

we get

$$2N_{2,4} + 5N_{5,1} = N_{1,5} + 2N_{1,2}N_{2,1}.$$

Knowing $N_{1,2} = 4$ and $N_{1,5} = 25$, e.g. by direct computation as in §7, we can solve the above equations and get $N_{2,1} = 1$ as well as

$$N_{2,4} = 14, \qquad N_{3,3} = 9, \qquad N_{4,2} = \frac{7}{2}, \qquad N_{5,1} = 1.$$

6. Higher genus and q-refined invariants

For an effective curve class $\underline{\beta}$ of X let $\beta^g_{p,q}$ be the class of stable log maps to X of genus g, class $\underline{\beta}$ and two marked points with contact orders p and q with D. The moduli space $\mathscr{M}(X,\beta^g_{p,q})$ of basic stable log maps of class $\beta^g_{p,q}$ has virtual dimension $g+1$. We cut this dimension down to zero by fixing the image of the first marked point and inserting a *lambda class*. Let $\pi : \mathcal{C} \to \mathscr{M}(X,\beta^g_{p,q})$ be the universal curve, of relative dualizing sheaf ω_π. Then $\mathbb{E} = \pi_\star\omega_\pi$ is a rank g vector bundle over $\mathscr{M}(X,\beta^g)$, called the Hodge bundle. The lambda classes are the Chern classes of the Hodge bundle, $\lambda_j = c_j(\mathbb{E})$. Let $\mathrm{ev} : \mathscr{M}(X,\beta^g_{p,q}) \to D$ be the evaluation map at the marked point of order p. Define the 2-marked log Gromov-Witten invariant

$$N^g_{p,q}(X,\beta) = \int_{[\![\mathscr{M}(X,\beta^g_{p,q})]\!]} (-1)^g\lambda_g \mathrm{ev}^\star[\mathrm{pt}] \in \mathbb{Q}.$$

Definition 6.1. Let $h : \Gamma \to B$ be a tropical curve. For a trivalent vertex V with multiplicity m_V (Definition 4.20) define, with $\boldsymbol{q} = e^{i\hbar}$,

$$m_V(\boldsymbol{q}) = \frac{1}{i\hbar}\left(\boldsymbol{q}^{m_V/2} - \boldsymbol{q}^{-m_V/2}\right).$$

For a vertex with higher valvency define $m_V(\boldsymbol{q}) = \prod_{V'\in V(h'_V)} m_{V'}(\boldsymbol{q})$ with h'_V as in Definition 4.20. For a bounded leg E with weight w_E define

$$m_E(\boldsymbol{q}) = \frac{(-1)^{w_E}}{w_E} \cdot \frac{i\hbar}{\boldsymbol{q}^{w_E/2} - \boldsymbol{q}^{-w_E/2}}.$$

Then define the ***q**-refined multiplicity* of h to be

$$m_h(\boldsymbol{q}) = \frac{1}{|\mathrm{Aut}(h)|} \cdot \prod_{V\in V(\Gamma)} m_V(\boldsymbol{q}) \cdot \prod_{E\in L_\Delta(\Gamma)} m_E(\boldsymbol{q}).$$

Theorem 6.2. *Let P be a point in an unbounded chamber of $\mathscr{S}_{p+q}$. Then, with $\boldsymbol{q} = e^{i\hbar}$,*

$$\sum_{g\geq 0} N^g_{p,q}(X,\beta)\hbar^{2g} = \sum_{h\in\mathfrak{H}^\beta_{p,q}(P)} m_h(\boldsymbol{q})$$

Proof. Consider a stable log map in $\mathscr{M}(X,\beta^g)$ and let $h : \Gamma \to B$ be its tropicalization. The genus of h is $g_h = g_\Gamma + \sum_V g_V$, where g_Γ is the genus of the graph Γ and g_V is the genus attached to a vertex V. Using gluing and vanishing properties of lambda classes, Bousseau showed in [Bou1] that Γ is still a tree ($g_\Gamma = 0$), i.e., all contributions to g_h come from vertices. Hence, h maps to an element of $\mathfrak{H}_\beta$ by forgetting genera at vertices g_V. So we can sum over $\mathfrak{H}_\beta$ but have to consider $\boldsymbol{q}$-refined contributions of vertices. By [Bou1], Proposition 29, the contribution of a vertex V with classical multiplicity m_V is $m_V(\boldsymbol{q})$. By [Bou2], Proposition 5.1, the contribution of a bounded leg L with weight w_L is $m_L(\boldsymbol{q})$. □

To obtain a higher genus version of Theorem 1, we have to $\boldsymbol{q}$-refine the slab functions in the initial wall structure $\mathscr{S}_0$. For $\boldsymbol{q}$-refined wall structures it turns out to be more conventient to work with the logarithm of such functions. Define the $\boldsymbol{q}$-refined initial wall structure $\mathscr{S}_0(\boldsymbol{q})$ to have the same slabs as $\mathscr{S}_0$ but with slab functions $f_{\mathfrak{p}} = 1 + z^{(m,0)}$ replaced by $f_{\mathfrak{p}}(\boldsymbol{q})$, where

$$\log\, f_{\mathfrak{p}}(\boldsymbol{q}) = \sum_{k\geq 1} \frac{(-1)^{k+1}}{k} \cdot \frac{i\hbar}{\boldsymbol{q}^{k/2} - \boldsymbol{q}^{-k/2}} z^{(km,0)}.$$

Note that the coefficient of $z^{(km,0)}$ is the $\boldsymbol{q}$-multiplicity of a bounded leg of weight k. Let $\mathscr{S}_\infty(\boldsymbol{q})$ be the consistent $\boldsymbol{q}$-refined wall structure obtained from $\mathscr{S}_0(\boldsymbol{q})$. Let P be a point in an unbounded chamber of $\mathscr{S}_{p+q}(\boldsymbol{q})$. Then a broken line $\mathfrak{b} \in \mathfrak{B}_{p,q}(P)$ has ending monomial $a_{\mathfrak{b}}(\boldsymbol{q}) t^{p+q} x^p$ for $x = z^{(-m_{\text{out}},-1)}$.

Definition 6.3. Define $\vartheta_q(\boldsymbol{q})$ and $\alpha^r_{p,q}(\boldsymbol{q})$ similarly to ϑ_q and $\alpha^r_{p,q}$ but with $\boldsymbol{q}$-refined broken lines.

Theorem 6.4 (Higher genus version of Theorems 1 and 2)**.**
Write $x = z^{(-m_{\text{out}},-1)}$ *and* $t = x^{(0,1)}$*. Then*

$$\vartheta_q(\boldsymbol{q}) = z^{-q} + \sum_{p=1}^{\infty} \sum_{g\geq 0} p N^g_{p,q} \hbar^{2g} t^{p+q} x^p \tag{3}$$

Moreover, $\alpha^r_{p,q}(\boldsymbol{q}) = 1$ *if* $r = p + q$ *and otherwise*

$$\alpha^r_{p,q}(\boldsymbol{q}) = \sum_{g\geq 0} ((p-r) N^g_{p-r,q} + (q-r) N^g_{q-r,p}) \hbar^{2g} t^{p+q-r}, \tag{4}$$

where we define $N^g_{p,q} = 0$ *whenever* $p \leq 0$*.*

Proof. We have $\log\, f_{\mathfrak{p}}(\boldsymbol{q}) = \sum_{w=1}^{\infty} \sum_{h\in\mathfrak{H}_{\mathfrak{p},w}} m_h(\boldsymbol{q}) z^{(wm_{\mathfrak{p}},0)}$, the $\boldsymbol{q}$-refined version of Proposition 4.24. Since breaking is similar to scattering (Proposition 2.2), at each breaking point $x \in \text{Int}(\mathfrak{p})$ of $\mathfrak{b}$ we pick up a factor of $m_{h_x}(\boldsymbol{q})$ for $h_x \in \mathfrak{H}^\circ_{\mathfrak{p},d}$ a tropical sub-disk of $h_{\mathfrak{b}}$. Together these factors exactly give $m_{h_{\mathfrak{b}}}(\boldsymbol{q})$, so we have a $\boldsymbol{q}$-refined version of Theorem 4.25: $a_{\mathfrak{b}}(\boldsymbol{q}) = m_{h_{\mathfrak{b}}}(\boldsymbol{q})$. Now the proof follows from the definitions of $\vartheta_q(\boldsymbol{q})$ and $\alpha^r_{p,q}(\boldsymbol{q})$. □

7. Example calculations

We use a sage code to calculate the numbers $N_{1,3d-1}(\mathbb{P}^2, dL)$ for $d \leq 4$. It can be found on timgraefnitz.com.

In the code, broken lines are implemented in reversed order. We start with point P on an unbounded maximal cell of $\mathscr{S}_d(\mathbb{P}^2)$ and a broken line coming out of this point in the negative of the unique unbounded direction m_{out}, with attached monomial $z^{qm_{\text{out}}}$. We can do this, since all broken lines that end in P have to be parallel to m_{out}. When the broken line hits a wall, we apply the transformation $z^m \mapsto f^{n\cdot m} z^m$, where n is the normal direction of the wall. Each term in $f^{n\cdot m}z^m$ gives a new possible broken line. The above procedure is applied recursively until the direction of the new broken line is m_{out}. Then we have found a broken line with asymptotic monomial $z^{qm_{\mathrm{out}}}$ and ending in P. If we add together the coefficients $a_{\mathfrak{b}}$ of the broken lines $\mathfrak{b}$ with asymptotic monomial $z^{qm_{\mathrm{out}}}$ and resulting monomial $a_{\mathfrak{b}}z^{-pm_{\mathrm{out}}}$ we get the tropical count $N_{p,q}^{\mathrm{trop}}(\mathbb{P}^2, dL)$, where $d = (p+q)/3$. Using the tropical correspondence (Theorem 4.22) we obtain the 2-marked log Gromov-Witten invariants $N_{p,q} = N_{p,q}(\mathbb{P}^2, dL)$. Figure 7.1 shows the broken lines the code produces for $d = 2$. This gives

$N_{1,5} = 25$	$N_{2,4} = 14$	$N_{3,3} = 9$	$N_{4,2} = \frac{7}{2}$	$N_{5,1} = 1$

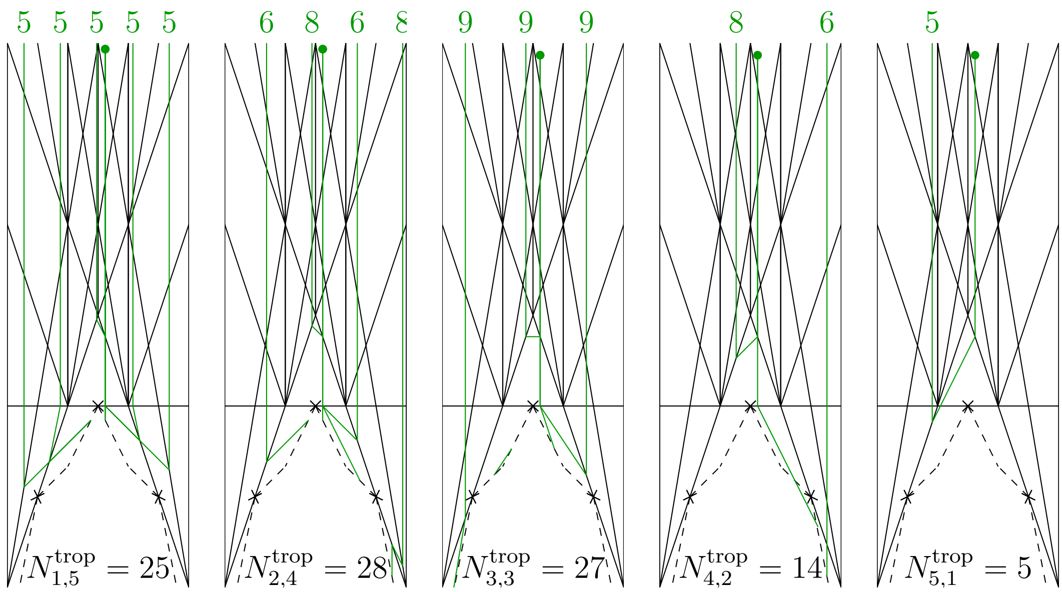


FIGURE 7.1. Broken lines used to compute $N_{p,q}(\mathbb{P}^2, 2L)$. The numbers above the infinite segments are the coefficients of the broken lines.

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